

Thermal noise limitations to force measurements with torsion pendulums: Applications to the measurement of the Casimir force and its thermal correction

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A general analysis of thermal noise in torsion pendulums is presented. The specific case where the torsion angle is kept fixed by electronic feedback is analyzed. This analysis is applied to a recent experiment that employed a torsion pendulum to measure the Casimir force. The ultimate limit to the distance at which the Casimir force can be measured to high accuracy is discussed, and in particular we elaborate on the prospects for measuring the thermal correction.

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I. INTRODUCTION

In a recent measurement of the Casimir force [1], a torsion pendulum was used. This measurement provided high accuracy measurements of this force at separations greater than $1\ \mu\text{m}$ [2]. This is because the torsion pendulum is among the most sensitive physical measurement devices, as has been recognized since Cavendish's measurement of the gravitational constant G over two centuries ago [3].

The limitations to force measurements due to thermal energy in the torsion mode, have been long known [4] and extensively studied. We address the particular case (as in [2]) where electronic feedback is employed to keep the angle fixed. The interest in Casimir force measurements suggests that we present the results of our analysis of the experiment described in [2].

Further measurements of the Casimir force, at large plate separation, are motivated by recent theoretical work that suggests a large correction to the thermal contribution to the Casimir force [5]. This work is not universally accepted [6] and it has been suggested that the correction is material dependent [7]. The specific point of [7] is that both [5] and [6] are correct, respectively, for dielectrics and for metals, and the respective regions of validity are addressed in [7]. The theory put forward in [7] can be tested with a poorly conducting material such as lightly doped germanium where the electron mean free path is less than the electromagnetic skin depth in the material. In this case, the analysis presented in [5] is likely applicable. For a metal film, as used in [2], the analysis presented in [6] is applicable, although the general validity of this approach has been questioned [8]. A comparison of the Casimir force for metals, semiconductors, and dielectrics at distances greater than $4\ \mu\text{m}$, with 10% level accuracy, is necessary to test the assertion put forward in [7]. Such measurements would provide data to help resolve the controversy surrounding the large thermal correction. This controversy is discussed at length in a recent review by Milton [9].

In this paper, we review the calculations of thermal noise in force measurements for free torsion pendulums. Previously, Newman [10] discussed three modes of operation of a torsion pendulum; here we analyze a fourth mode, where magnetic or electrostatic feedback is used to measure the

force. Magnetic feedback was employed in a demonstration experiment to measure the gravitational constant [11], while this technique was slightly modified in [2], where electrostatic feedback was employed to eliminate nonlinear magnetic and hysteresis effects.

II. ANGLE NOISE IN A FREE TORSION PENDULUM

We consider the case of a torsion pendulum where a mass m with moment of inertia I is suspended in the Earth's gravitational field by a fiber (or wire) with angular restoring torque $\tau = -\alpha\theta$ at an absolute temperature T . Generally, the thermal angular fluctuations for the swinging (gravitational) pendulum modes are vastly smaller than the torsional angular fluctuations. Because each mode has $kT/2$ of thermal energy, the rms angular fluctuations are [4]

$$\delta\theta = \sqrt{\frac{kT}{\alpha}} \gg \sqrt{\frac{kT}{mg\ell}}, \quad (1)$$

where k is Boltzmann's constant, g is the gravitational acceleration, and ℓ is the pendulum length. For $\alpha = 1\ \text{dyn cm/rad}$, $m = 100\ \text{g}$, and $\ell = 10\ \text{cm}$, the swinging mode thermal noise amplitude is about three orders of magnitude smaller than the torsion mode noise.

The case of the free torsion pendulum is when there is no external driving or restoring torque applied to the system. The angular fluctuations, through the fluctuation-dissipation theorem [12] and the Langevin equation [13], are described by

$$I\ddot{\theta} + \gamma\dot{\theta} + \alpha\theta = \tau(t) \Rightarrow \quad (2)$$

$$(-I\omega^2 + i\gamma\omega + \alpha)\theta_\omega = \sqrt{4kT\gamma\Delta f}, \quad (3)$$

where $\tau(t)$ is the thermal fluctuation torque on the torsion pendulum, and θ_ω is the fluctuation amplitude at frequency ω in a bandwidth Δf Hz.

The mean-square spectral density of the fluctuations is, therefore,

$$\frac{|\theta_\omega|^2}{\Delta f} = \frac{4kT\gamma}{\alpha^2} \frac{1}{(1-x^2)^2 + Q^{-2}x^2}, \quad (4)$$

where

$$x = \frac{\omega}{\omega_0}, \quad \omega_0^2 = \frac{\alpha}{I}, \quad (5)$$

and the mechanical quality factor is

$$Q = \frac{\alpha}{\omega_0 \gamma}. \quad (6)$$

To determine the variance of a series of measurements of θ we need the autocorrelation function $R_\theta(t)$. This can be determined by the inverse of the Wiener-Khinchin theorem [14],

$$R_\theta(t) = \text{Re} \left[\int_0^\infty \frac{|\theta_\omega|^2}{\Delta f} e^{i\omega t} df \right], \quad (7)$$

where Re means the real part and $\omega = 2\pi f$. This integral can be calculated by contour integration methods by noting that the denominator can be factored as

$$(1 - x^2)^2 + Q^{-2}x^2 = (x - \Omega + iQ^{-1}/2)(x + \Omega + iQ^{-1}/2) \\ \times (x - \Omega - iQ^{-1}/2)(x + \Omega - iQ^{-1}/2), \quad (8)$$

where

$$\Omega = \sqrt{1 - \frac{Q^{-2}}{4}}. \quad (9)$$

Noting that $|\theta_\omega|^2 = |\theta_{-\omega}|^2$ and $df = \omega_0 dx / 2\pi$, the integration path is taken from $x = -\infty$ to $x = +\infty$, and the contour is closed by a semicircle in the upper half plane where the poles are located at $\pm\Omega + iQ^{-1}/2$ so that $\lim_{t \rightarrow \infty} R_\theta(t) = 0$. We thus find

$$R_\theta(t) = \text{Re} \left[\frac{4kT\gamma}{\alpha^2} \frac{\omega_0}{2\pi} \frac{\pi}{4} e^{-\omega_0 t/2Q} [2Q \cos(\Omega\omega_0 t) \\ + \Omega^{-1} \sin(\Omega\omega_0 t)] \right]. \quad (10)$$

Recalling that $\alpha/\omega_0\gamma = Q$,

$$R_\theta(t) = \frac{kT}{\alpha} e^{-\omega_0 t/2Q} \text{Re}[\cos(\Omega\omega_0 t) + (2\Omega Q)^{-1} \sin(\Omega\omega_0 t)]. \quad (11)$$

The integral of the spectral density is $R_\theta(0)$ and should be equal to the mean-square fluctuations,

$$R_\theta(0) = \frac{kT}{\alpha}, \quad (12)$$

which is true for Eq. (11) and is an important check for the autocorrelation function.

The general operating procedure is to take many closely time-spaced measurements of the torsion pendulum angle, with the time spacing less than any correlation time in the system. The variance of the sample means, for many closely spaced measurements over a time interval T_m , assuming without loss of generality, that the mean is zero, is [15]

$$\sigma^2(\theta) = \frac{2}{T_m} \int_0^{T_m} \left(1 - \frac{t}{T_m}\right) R_\theta(t) dt. \quad (13)$$

In the limit of a very long measurement time, the second term in the integral vanishes compared to the first, and in the limit of large Q , we find

$$\sigma^2(\theta) = \frac{2kT}{\alpha} \frac{1}{Q\omega_0 T_m}. \quad (14)$$

Noting that

$$Q = \frac{\pi\tau_d}{\tau_0} = \frac{\omega_0\tau_d}{2} = \frac{I\omega_0}{\gamma} = \frac{\alpha}{\omega_0\gamma}, \quad (15)$$

where τ_0 is the oscillation period and τ_d is the $1/e$ amplitude damping time, the variance can be recast in the form

$$\delta\theta_{\text{rms}} = \sqrt{\frac{4kT}{I\tau_d T_m} \left(\frac{\tau_0}{2\pi}\right)^2}, \quad (16)$$

which is a smaller by factor of $\sqrt{2/3}$ than the result given in [16], and quoted in [17] as Eq. (10).

The force measurement noise is simply given by, using Eq. (14) and (6),

$$\delta F_{\text{rms}} = \frac{\alpha\delta\theta_{\text{rms}}}{R} = \sqrt{\frac{2\gamma kT}{R^2 T_m}}, \quad (17)$$

where R is the effective radius where the force is applied. We thus see that R should be as large as possible, γ as small as possible, and that the sensitivity is independent of α and I .

III. TORSION PENDULUM WITH ELECTRONIC FEEDBACK

In the experiment described in [2], feedback was used to keep the torsion pendulum angle fixed in space. The restoring torque was generated by changing the voltage between a fixed plate and the conducting pendulum body, which was grounded. The force between capacitor plates with a potential difference V is proportional to V^2 . However, because the feedback signal was added to a relatively large fixed voltage V_0 , the system was linear in that $V^2 = (V_0 + \delta V)^2 \approx V_0^2 + 2V_0\delta V$.

A simple proportional-plus-integral feedback scheme was used in [2]. For this system, Eq. (2) becomes

$$I\ddot{\theta} + \gamma\dot{\theta} + \alpha\theta = \tau(t) - \beta\theta - \kappa \int \theta dt, \quad (18)$$

where β and κ are the proportional and integral feedback gain, respectively. The experiment [2] was operated in the limit of very large damping, which means that the inertial ($I\ddot{\theta}$) term can be neglected. The system was in some ways equivalent to a phase-locked loop (PLL) because in this limit $\dot{\theta}$ is a constant value proportional to the feedback signal [18], and the standard PLL analysis techniques can be applied to this problem. The spectral amplitude can be determined as before,

$$(i\gamma\omega + \alpha)\theta_\omega = \left(-\beta - \frac{\kappa}{i\omega}\right)\theta_\omega + \sqrt{4kT\gamma\Delta f} \quad (19)$$

and the spectral density is

$$\frac{|\theta_\omega|^2}{\Delta f} = \frac{4kT\gamma}{\kappa^2} \frac{\omega^2}{(1-x^2)^2 + Q^{-2}x^2}, \quad (20)$$

where

$$x = \frac{\omega}{\omega_0}, \quad \omega_0^2 = \frac{\kappa}{\gamma}, \quad Q = \frac{\kappa}{\omega_0(\alpha + \beta)}. \quad (21)$$

For this experiment, the output of the integrator provided the torque measurement, and

$$\frac{R^2|F_\omega|^2}{\Delta f} = \frac{\kappa^2|\theta_\omega|^2}{\omega^2\Delta f}, \quad (22)$$

which is equal to Eq. (4) multiplied by α^2 , but with different meanings for the parameters. We can immediately write down the force autocorrelation as measured at the integrator output

$$R_F(t) = \frac{\omega_0\gamma kT}{R^2} e^{-\omega_0 t/2Q} [Q \cos(\Omega\omega_0 t) + (2\Omega)^{-1} \sin(\Omega\omega_0 t)]. \quad (23)$$

The response time is optimum when $Q=0.5$ so that the system is critically damped. In this case it is easy to calculate the rms noise after many closely spaced measurements as before

$$\delta F_{\text{rms}} = \sqrt{\frac{2\gamma kT}{R^2 T_m}}, \quad (24)$$

which is the same result as before, Eq. (17).

A. Electronic noise

So far in this analysis, electronic noise has been neglected. This is reasonable because the differential capacitor signal used in [2] to provide a measure of the torsion angle had an intrinsic sensitivity of 100 V/rad. Given a typical integrated operational amplifier noise of tens of n V/ $\sqrt{\text{Hz}}$ at the ac bridge frequency of 4.2 kHz (the amplifier noise voltage is comparable to the Johnson noise in the 1 M Ω bridge resistors), the electronically limited angular resolution was of order n rad/ $\sqrt{\text{Hz}}$, much below the intrinsic thermal angular fluctuations of the torsion fiber, which, from Eq. (1) were of order μrad for $\alpha \approx 1$ dyn cm/rad.

B. Experimental examples

Many recent measurements that have used torsion pendulums have employed fine tungsten wires as the support and torsion element. The tensile strength of tungsten is among the highest of known materials, so a very fine diameter wire can be used to support a substantial mass. Both the damping coefficient γ_W and the torque coefficient α depend on the radius to the fourth power [19]

$$\gamma_W = \frac{\eta \pi r^4}{2 \ell}, \quad \alpha = \frac{Z \pi r^4}{2 \ell}, \quad (25)$$

where r is the wire radius, the internal viscosity $\eta=9.37 \times 10^9$ poise for tungsten ([19], Table Chap. V, Table III), and the tangential coefficient of elasticity (or torsion modulus) $Z \approx 1.8 \times 10^{12}$ dyn/cm², and depends on the wire diameter ([20], Fig. B 3.2–17). The values of α for two experiments [2,17] are in reasonable agreement with this formula.

The experiments [2,17] were operated with higher damping than that intrinsic to tungsten wire, γ_W . For [2], $\gamma_W = 4.7 \times 10^{-2}$ dyn cm sec/rad, compared to the experiment $\gamma = 10$ dyn cm sec/rad as set by the magnetic damping. For [17], $\gamma_W = 1.8 \times 10^{-4}$ dyn cm sec/rad, compared to the experiment $\gamma = 0.2$ dyn cm sec/rad due to background gas. Both of these experiments relied on extra damping to reduce the effects of perturbations that drove the various pendulum modes of the system, and this extra damping effectively increased the system bandwidth.

As we have shown in this paper, the only parameter that affects the force measurement sensitivity due to thermal fluctuations is the mechanical damping coefficient γ . For the two experiments discussed here, the wire length could have been substantially shortened without increasing γ significantly, e.g. for [2] the intrinsic torsion wire damping becomes equal to the magnetic damping when $\ell = 0.4$ cm, and for [17], equal to the residual gas damping when $\ell = 2.4$ cm. From a thermal noise standpoint, there is very little gain in having the wire length larger than these values.

The effects of tilt depend linearly on wire length, and tilt was a major noise source in [2] where, for short-term noise, thermal effects dominated. However, the $1/f$ noise corner was around 5 Hz, and the effective system noise was about 100 times larger than the expected thermal noise in the wire for measurements periods of 10 s separated by 20 s. In this case it was evident that the tilt was the largest noise factor; distortion of the concrete floor by the weight of the experimenter was a significant factor in the alignment of the apparatus, and this floor was directly coupled to the building. For [17], the system noise was about 10 times the thermal noise. Effects of external noise due to tilt of the apparatus could have been reduced, in a practical sense, by reducing the torsion wire length by an order of magnitude in these experiments without a significant loss of sensitivity.

IV. CONCLUSION: APPLICATION TO CASIMIR FORCE MEASUREMENTS

We have shown that the single parameter that determines the thermal force noise after a long-time average of a torsion pendulum signal is the mechanical damping coefficient γ . Several experiments that have been performed to date employed extra system damping to shorten the time required to recover from larger perturbations, and thereby effectively increase the system bandwidth. This implies that many experiments could have employed a shorter torsion wire, which would have substantially reduced the noise due to tilting of the apparatus.

With electrical feedback, the system Q can be arbitrarily tuned, and in particular $Q=0.5$, corresponding to critical

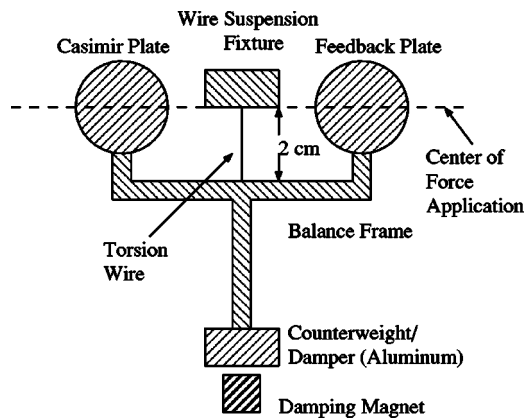


FIG. 1. In addition to shortening the wire length in a torsion pendulum, placing the centers of the electrostatic feedback and Casimir plates at the point of suspension of the torsion wire further reduces the effects of apparatus tilt.

damping, can be obtained. Operating with this Q provides the optimum measurement bandwidth [18]. The thermal noise is still determined by the intrinsic pendulum γ [see Eq. (24)]. The implication is that the Q can be made small, as is desirable from bandwidth consideration (e.g., so that the system achieves equilibrium in a shorter time after the force is changed). A high bandwidth allows the possibility of a measurement frequency higher than the inevitable system $1/f$ noise corner frequency.

A short torsion wire (a few centimeters) suggests a new configuration for a pendulum. As shown in Fig. 1, the “centers of force measurement” at the test bodies can be located at the same height as the support point of the torsion wire. Then if the pendulum tilts, the change in separation between the plates (in a Casimir measurement) is second order in the tilt angle. For flat plates, there is only a change in separation when the tilt is perpendicular to the axis formed by the line between the test bodies. For curved plates, tilts around either axis lead to a second order in angle change in separation. With such a configuration, it should be possible to reduce the tilt noise by a factor of 100, which would allow measurement at level determined by the intrinsic thermal noise. It is tempting to consider contouring the plates so that there is no

change in plate separation with tilt angle, however, this possibility is likely impractical.

The thermal correction to the Casimir force, between curved perfectly conducting surfaces with net radius of curvature \mathcal{R} , is [21]

$$F(d) \approx \frac{2.4\mathcal{R}kT}{4d^2}, \quad (26)$$

where d is the distance of separation between the curved surfaces, and this equation is accurate for $d > 4 \mu\text{m}$, where the usual zero-point Casimir force is relatively small. The result in [5] indicates a factor of two reduction in the force for this or larger separations. The same conclusion was reached by Høye *et al.* [22]. In order to test this theory, a 10% measurement accuracy is needed at $d \approx 4 \mu\text{m}$, or $\delta F = 0.1 \mu\text{dyn}$ for $\mathcal{R} = 10 \text{ cm}$. This level of sensitivity is 10 times the intrinsic thermal noise of [2], so with the control of tilt noise, a relevant test of the theory [5] should be possible.

The idea put forward in [7] is to employ materials with different electron mean-free-paths compared to the electromagnetic skin depth, for mode frequencies that contribute significantly to the Casimir force. The analysis [5] is only applicable for materials where the skin depth is greater than the mean free path, otherwise nonlocal correlations between currents in the material and the electromagnetic field exist, and the description of the field propagation into the material in terms of a wave vectors is not complete. However, it has been suggested that the problem for good conductors can be calculated from a surface impedance approach; the electric and magnetic fields at the plate surfaces can always be linearly related, and this relationship can be used to determine the electromagnetic modes frequencies between the plates and thus the attractive force [6,7]. Therefore it would be useful to compare measurements of the Casimir force at larger separations for both good conductors (Au or Cu) and for poor conductors (Ge) where the skin depth greatly exceeds the electron mean free path. Germanium infrared optical lenses are available with \mathcal{R} of order 10 cm [23], and these can be used as the plates in a Casimir force measurement using a torsion pendulum as in [2].

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